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Polynomial families and Schrödinger equation: one example for nonhypergeometric type of correspondence

George Krylov[1,2] **and Marko Robnik**[1]

[1] Centre for Applied Mathematics and Theoretical Physics, University of Maribor, Krekova 2, SI-2000 Maribor, Slovenia
[2] Department of Physics, Belarusian State University, Fr. Skaryna av. 4, 220050 Minsk, Belarus

E-mail: george.krylov@uni-mb.si and robnik@uni-mb.si

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Abstract
We discuss the explicit construction of the Schrödinger equations admitting representation through some family of general nonorthogonal polynomials. The specific choice of the third-order polynomial coefficient functions, that lead to quasi-solvable families of Schrödinger potentials, is considered in detail.

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1. Introduction

The known exactly solvable potentials in one-dimensional quantum problems, including those emerging from the separable potentials (see e.g. [1–3] and references therein) allow us to construct eigenfunctions for bound states based on the requirement of termination for some hypergeometric series. The resulting polynomial families are of great importance and typically are known in great and fine detail (see e.g. [3]). Recently we proposed the unified approach for the search for the Schrödinger equations possessing polynomial solutions [4], which contains and extends in some aspects the Natanzon construction of solvable quantum potentials [5,6] as well as Turbiner's consideration [7] of the generalized Bochner problem. One of the earliest important papers on solubility in quantum mechanics was [8]; the subsequent flow is enormous and separated on the following main approaches.

Supersymmetric quantum mechanics (SUSY QM), having much in common with the known Infeld–Hull factorization technique [9], but developed originally in a different context in [10], gives much deeper insight into the solvability problem than ever before. In the first stage of the development its significance seems to be actually overestimated, especially when it was found that not only can series of well known solvable potentials be easily investigated using it, but the supersymmetric WKB (SWKB) approximation leads to exact results for spectra just in the semiclassical limit. The understanding of the limits of SWKB obtained in [11] clarify the situation and, surprisingly, lead to an increase of the interest in SUSY QM and especially to its generalization (for a review see e.g. [12]; some interesting new results on the topic can be found in [13–18]).

The second type of approach, developed separately, though close and becoming closer and closer to SUSY QM, is numerous algebraic approaches, among which we would like to mention relatively old classical papers [19, 20] with extensive reference lists therein and some recent ones [21–24] discussing different aspects of the problem.

Finally, the analytical ways of attacking the problem (that we also adhere to) are also very numerous, so we only cite a few recent papers [25–27], demonstrating some new ideas but still being away from the topics discussed in this paper.

The basic idea of the present work is that, similar to [7], we construct some general form of ODEs (or PDEs) which has the polynomial solutions in a natural way due to its specific though sufficiently general form. We then construct the Schrödinger equation associated with the original equation for the polynomials using a number of transformations (point canonical transformation and similarity transformation in the one-dimensional case).

It turns out that in this way some interesting questions and problems appear, which we shall demonstrate in some specific examples in this paper. We will concentrate on some quasi-exactly solvable (QES) problems, considering the method of regular construction of QES potentials within a given family, and demonstrate the specific feature of our approach.

Though QES quantum problems are well known today [18, 28–33] (see also [34] and references therein), systematically investigated starting from [35], there is no general classification scheme for QES systems, and fundamental reasons leading to quasi-exact solvability remain unclear. One more thing which remains mysterious is the orthogonality properties of the polynomial families associated with QES problems. The latter were investigated in early papers [32, 33, 35], considered as a nonclassical orthogonal family of discrete variable E (Bender–Dunne polynomials), whereas in the later paper [29] the absence of orthogonality for some specific cases was demonstrated. It is worthwhile to mention that an additional symmetry arising in the considered cases, such as investigated in [36], can influence significantly and sometimes unexpectedly the spectral properties of a system.

Therefore, a systematic approach to the construction and classification of QES problems would be of great interest and it will be proposed hereafter with the discussion of some explicit examples.

The general form of the second-order linear differential equations (SODEs) allowing polynomial solutions at some specifically chosen values of their coefficients is [4]

$$\hat{\mathcal{L}}_k y(x) = P_{k+2}(x) y''(x) + Q_{k+1}(x) y'(x) + R_k(x) y(x) = 0 \qquad (1)$$

where we introduce an index (k) for the operator to stress that this is the finite-dimensionally generated operator family, fully defined by the choice of the coefficient functions $P_{k+2}(x)$, $Q_{k+1}(x)$ and $R_k(x)$, which are polynomials of order $k+2$, $k+1$ and k, respectively.

It is easily understood that differential operator $\hat{\mathcal{L}}_k$ maps the space of the nth-order polynomials $F_n[x]$ into the space $F_{n+k}[x]$ (see [4, 7] for more details). As both spaces are finite dimensional, the condition of nontrivial kernel Ker $\hat{\mathcal{L}} \neq 0$ leads simply to a linear algebraic problem for operator representation in this space plus k additional conditions imposed on the coefficients of coefficient functions. In terms of [7] the last means that such conditions are those when the operator $\hat{\mathcal{L}}[k]$ simply preserves $F_n[x]$ (flag preserving conditions).

For $k = 0$ we have only the single condition of vanishing determinant of the matrix corresponding to the operator $\hat{\mathcal{L}}$ in the basis of the monomials x^i, $i = 0 \ldots n$. This is just the standard eigenvalue (Sturm–Liouville) problem [4]. If $k > 0$, as we will see, the situation is more interesting. We want to discuss here only the case $k = 1$ in detail, so that the appropriate polynomial coefficients have at most third order. The consequences for higher k will be briefly discussed in relation to the orthogonality problem only (see below).

We are going to demonstrate, and this will be the main goal of the paper, that the situations with solvability for exactly solvable potentials and QES ones are in some sense complementary to each other. The Schrödinger equation (and therefore quantum potential) for exactly solvable cases can be associated with a single polynomial family in such a way that the ith excited state (bound) eigenfunction is constructed through the ith-order polynomial of a given family [3] having exactly i zeros (nodal points), typically as a finite hypergeometric series.

In QES cases, it turns out that we can construct a quantum potential having precisely n known eigenstates all expressed through i different nth-order polynomials having $0, 1 \ldots i$ roots in the real domain, so that the wavefunction possesses the necessary properties (the ith excited state wavefunction for the one-dimensional problem, according to the Sturm–Liouville theory, must have precisely i roots in the domain) [2].

Let us start with the construction of the polynomial solutions for the equation (1).

2. QES quantum problems through polynomial ansatz for flag preserving differential equations

Let us consider the equation (1) with $k > 0$ as a basic one, possessing polynomial solutions in the form of the nth-order polynomial

$$y_n(x) = \sum_{i=0}^{n} c_i x^i. \tag{2}$$

Substituting and equating to zero coefficients at all powers of x, we have $n + k + 1$ coefficients at different power of x to be equal to zero, whereas the number of unknown coefficients c_i is evidently $n + 1$. To implement the demands we have to impose k additional conditions on the coefficients of polynomial P, Q, R (say p_i, q_i, r_i), that correspond to the requirement for the matrix of the linear system to have the range equal to n.

The above condition related to the topmost power (x^{n+k}) simply reads

$$n(n-1)p_{k+2} + nq_{k+1} + r_k = 0 \tag{3}$$

whilst the rest are a little more complicated, though they could be written in a closed form for some explicit choice of k and n.

Now, we want to transform the equation (1) to the form of the Schrödinger equation. This could be done as in [4] via a pair of transformations: the first one is a point canonical transformation (variable change); the second one is a gauge transformation.

First, we make the variable change

$$\begin{aligned} & x = F(u) \\ & \frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{F'(u)}\frac{\mathrm{d}}{\mathrm{d}u} \\ & \frac{\mathrm{d}^2}{\mathrm{d}x^2} = \frac{1}{F'^2(u)}\frac{\mathrm{d}^2}{\mathrm{d}u^2} - \frac{F''(u)}{F'^3(u)}\frac{\mathrm{d}}{\mathrm{d}u} \end{aligned} \tag{4}$$

and introduce some as yet undefined but prescribed function of new coordinate $\omega(u)$ (which we could define later for the sake of the most convenient choice)

$$\omega^2(u)[F'(u)]^2 = P_{k+2}(x) \tag{5}$$

to obtain

$$\omega(u)^2\, y''(u) + R_k(F(u))y(u) + \frac{\omega(u)\, y'(u)\, \left(2\, Q_{k+1}(F(u)) + 2\sqrt{P_{k+2}(F(u))}\,\omega'(u) - P'_{k+2}(F(u))\right)}{2\sqrt{P_{k+2}(F(u))}} = 0. \quad (6)$$

Now we perform a similarity transformation $Y(u) = \exp(\chi(u))y(u)$ and choose $\chi(u)$ in such a way as to kill the first-derivative term. This means the condition

$$\frac{Q_{k+1}(F(u))\,\omega(u)}{\sqrt{P_{k+2}(F(u))}} + \omega(u)\,\omega'(u) - \frac{\omega(u)\,P'_{k+2}(F(u))}{2\sqrt{P_{k+2}(F(u))}} = -2\omega(u)^2\chi'(u) \quad (7)$$

holds, that defines the gauge up to a constant.

Then we obtain the Schrödinger type equation in the form

$$\begin{aligned} Y''(u) + Y(u)\Bigg[&\frac{R_k(F(u))}{\omega(u)^2} - \frac{Q_{k+1}(F(u))^2}{4\,P_{k+2}(F(u))\,\omega(u)^2} + \frac{\omega'(u)^2}{4\,\omega(u)^2} \\ &- \frac{\omega''(u)}{2\,\omega(u)} - \frac{Q'_{k+1}(F(u))}{2\,\omega(u)^2} + \frac{Q_{k+1}(F(u))\,P'_{k+2}(F(u))}{2\,P_{k+2}(F(u))\,\omega(u)^2} \\ &- \frac{3\,P'_{k+2}(F(u))^2}{16\,P_{k+2}(F(u))\,\omega(u)^2} + \frac{P''_{k+2}(F(u))}{4\,\omega(u)^2}\Bigg] = 0. \end{aligned} \quad (8)$$

The problem of the solvability of the Schrödinger equation (8) now can be formulated as follows:

Can some fractions in (8) be rewritten to obtain a free parameter that could be interpreted as an 'energy parameter'?

The simple answer is evidently 'yes', for example by choosing $\omega(u) = 1$ and keeping nonzero free coefficient in R_k ($r_0 \neq 0$), though some more complicated schemes may be of interest (see [4] and the appropriate discussion for the solvable case therein).

It is worthwhile to mention here that the case where there is no free parameter that could be interpreted as E, which corresponds to the 'zero-energy' solution, usually does not correspond to an eigenstate of the system (in contrast to the situation for a diffusion equation, where the eigenstate $E = 0$ always exists).

Now we can consider some explicit examples to see what could be obtained in this way.

3. Construction for third-order polynomial coefficient functions

We start from equation (1) (with $k = 1$) and assume all coefficients are real. As the third-order polynomial has at least one real root, without loss of generality we can put it equal to zero, because the change of independent variable $x \to x - x_0$ leads to the same form of the equation but with modified coefficients of the same order. So, allowing that other roots for polynomials could be complex we rewrite (1) in a more convenient form for subsequent consideration (the topmost coefficient in $P_3(x)$ can be always chosen as unity)

$$x(x - x_1)(x - x_2)y''(x) + \alpha(x - x_3)(x - x_4)y'(x) + (\beta x + \gamma)y(x) = 0 \quad (9)$$

with x_1, x_2, x_3, x_4 being the roots of our coefficient polynomials.

Now, as already done, we are looking for the solution of (9) in the form of an nth-order polynomial

$$y_n(x) = \sum_{i=0}^{n} c_i x^i \quad (10)$$

leading to the only condition to hold:

$$n(n-1)+n\alpha+\beta=0. \tag{11}$$

The remaining equations are simply homogeneous linear equations for $n+1$ coefficients c_i. The condition of a nontrivial solution is the condition of vanishing of the determinant of the following three-diagonal matrix with nonzero elements expressed as ($i=1,\ldots,n$):

$$M_{i,i}=\gamma-(\mathrm{i}-1)(x_3+x_4)\alpha-(\mathrm{i}-2)(\mathrm{i}-1)(x_1+x_2) \tag{12}$$

$$M_{i-1,i}=(\mathrm{i}-1)(\mathrm{i}-2)x_1x_2+(\mathrm{i}-1)x_3x_4\alpha \tag{13}$$

$$M_{i+1,i}=(\mathrm{i}-1)(\mathrm{i}-2)+(\mathrm{i}-1)\alpha+\beta \tag{14}$$

that leads, as is easy to see, to the following three-term recurrence relation for the determinant D_i, $i=1,\ldots n$:

$$\begin{aligned}D_{i+1}(\gamma)&=D_i(\gamma)(\gamma-\mathrm{i}(x_3+x_4)\alpha-\mathrm{i}(\mathrm{i}-1)(x_1+x_2))\\&\quad-D_{i-1}(\gamma)\left((\mathrm{i}-1)(\mathrm{i}-2)+(\mathrm{i}-1)\alpha+\beta\right)(\mathrm{i}x_3x_4\alpha+\mathrm{i}(\mathrm{i}-1)x_1x_2)\end{aligned} \tag{15}$$

with $D_1=\gamma$ and $D_2=-\gamma((x_3+x_4)\alpha+\gamma)-\beta x_3x_4\alpha$.

It is easy to show that these ith-order polynomials D_i as functions of γ cannot be related to the classical orthogonal polynomial families (though in this specific case they have much in common with Bender–Dunne polynomials [35]). Indeed, let us assume the opposite, then the classical Rodrigues' formula for the orthogonal polynomial family with the weight $W(x)$ reads [3]

$$p_i(x)=\frac{a_n}{W(x)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i\left(g^i(x)W(x)\right) \tag{16}$$

with $g(x)$ being a polynomial independent of i. We can use this formula for the two lowest polynomials D_i and obtain the system of two differential equations defining unknown functions $g(x)$ and $W(x)$. Then we have

$$\begin{aligned}&g(x)\,W'(x)+W(x)(\,g'(x)-x)=0\\&2\,g(x)\,W(x)\,g''(x)+2\,W(x)\,g'(x)^2+4\,g(x)\,g'(x)\,W'(x)\\&\qquad+g(x)^2\,W''(x)+(\alpha\beta x_3x_4-(x-\alpha(x_3+x_4))x)\;W(x)=0.\end{aligned} \tag{17}$$

The solution of the first equation (up to a nonessential constant multiplier) has the form

$$W(x)=g(x)^{-1}\exp\left\{\int\frac{x}{g(x)}\,\mathrm{d}x\right\} \tag{18}$$

that gives after its substitution into the second equation of (17) the following nonlinear SODE for $g(x)$:

$$g(x)g''(x)+x\,g'(x)+g(x)+\alpha\;((x_3+x_4)\,x+x_3\,x_4\,\beta)=0. \tag{19}$$

It is easy to see that (19) indeed has a polynomial solution, namely assuming $g(x)$ in the form $\sum_{i=0}^k g_ix^i$ we immediately obtain two possible solutions

$$g^{(1)}(x)=-\frac{3x^2}{2}+\alpha(x_3+x_4)\,x+\frac{\alpha\,\beta x_3x_4}{2} \tag{20}$$

$$g^{(2)}(x)=-\alpha\,\beta x_3x_4-\frac{(x_3+x_4)\,\alpha\,x}{2}. \tag{21}$$

The direct substitution would lead to a fairly complicated expression for arbitrary x_3, x_4 for a polynomial with the order higher than two, but to demonstrate the incorrectness of our initial hypothesis we simply obtain the explicit expression for the constructed family for the case we will investigate in the following, namely when $x_1=-x_2=\xi$ (symmetric case). Then, the first- and the second-order polynomials we constructed will coincide with D_1, D_2 as they should, but for the third- and fourth-order orthogonal polynomial family we have

$$p_3(x) = x^3 + \frac{3\,\alpha\,\beta\,x\,\xi^2}{7} \tag{22}$$

and

$$p_4(x) = x^4 + \frac{6\,\alpha\,\beta\,x^2\,\xi^2}{13} + \frac{3\,\alpha^2\,\beta^2\,\xi^4}{91} \tag{23}$$

whereas the direct calculation based on the recurrence relation gives for D_3 and D_4

$$D_3(x) = x^3 + (2\,\alpha^2\; + 3\,\alpha\,\beta)\xi^2 x \tag{24}$$

and

$$D_4(x) = x^4 + 2(\alpha + 3\beta + 3)\alpha\xi^2 x^2 + 3(2\alpha + \beta + 2)\alpha^2\beta\,\xi^4. \tag{25}$$

As one can see the third- and fourth-order polynomials D_i differ from those constructed with an assumption that they form a classical orthogonal polynomial family, which proves our statement.

The analytical solution of the three-term recurrence relation is just another problem, and for our purpose it will be sufficient to know that we can solve the appropriate characteristic equation, at least for small n.

At this point it is worthwhile to say a little more about the general case $k > 1$. As one can easily see from (1) and (9) it is just the $k = 1$ condition that leads to three-term recurrence relation (15), whereas for arbitrary k we have to obtain generally the $(k+2)$th-order recurrence relation. It is evident that we still have quasi-exact solvability properties, though the polynomials in this case are evidently not orthogonal with any weight function. This is precisely the result demonstrated in [29] for a particular case, which we automatically obtain based on rather general arguments.

Now, let us start with the transformation of the equation for polynomials to the Schrödinger equation and the investigation of topologically different cases.

4. Schrödinger equations associated with flag preserving operators for $k = 1$

We have to distinguish three regular cases, namely when (i) the polynomial A_3 in (9) has one root of order three, (ii) two different roots and (iii) three different roots.

Additionally, irregular cases are realized when the order of A_3 is less than three. We will not consider these here.

The first case (i) of interest is when all roots of A_3 are coincident, that is $A_3(x) = x^3$. Then the equation reads

$$x^3 y''(x) + \alpha(x - x_3)(x - x_4)y'(x) + (\beta x + \gamma)y(x) = 0. \tag{26}$$

In accordance with [4] we perform the point canonical transformation (variable change) $x = 4u^{-2}$, $\;y(x) \to y(u)$, and the similarity (gauge transformation) for the wavefunction $Y(u) = \exp(\chi(u))y(u)$ with

$$\chi(u) = -\frac{\alpha x_3 x_4 u^4}{64} + \frac{\alpha(x_3 + x_4)u^2}{8} + \left(\frac{3}{2} - \alpha\right)\,\log(u) \tag{27}$$

to obtain the Schrödinger equation of the form

$$Y''(u) + (\varepsilon - V(u))\,Y(u) = 0 \tag{28}$$

with the potential $V(u)$ in the following form:

$$V(u) = \frac{A}{u^2} + Bu^2 + Cu^4 + Du^6 \tag{29}$$

where we introduced the new parameters A, B, C, D and ε, which are expressed through the parameters α, β, x_3, x_4 as

$$A = \alpha^2 - 2\,\alpha - 4\,\beta + \tfrac{3}{4} \tag{30}$$

$$B = \frac{\alpha(\alpha(x_4^2 + 4x_3x_4 + x_3^2) - 6x_3x_4)}{16} \tag{31}$$

$$C = -\frac{\alpha^2 x_3x_4(x_3 + x_4)}{32} \tag{32}$$

$$D = \frac{\alpha^2 {x_3}^2 {x_4}^2}{256} \tag{33}$$

$$\varepsilon = \gamma + \frac{\alpha(\alpha - 2)(x_3 + x_4)}{2}. \tag{34}$$

As one can easily see the case when one of the roots x_3, x_4 is equal to zero immediately leads to the ordinary radial harmonic oscillator problem. It is nontrivial to discover this when looking at the initial equation rather than at the Schrödinger one. It occurs due to the existence of the common roots of $A_3(x)$ and $A_2(x)$. Indeed, it is possible to show, though we do not intend to do it here, as it will be the subject of a separate publication, that if the coefficient polynomial $A_{k+2}(x)$ shares a root (x_i) with $A_{k+1}(x)$, in the equation (1), the substitution $y(x) = (x - x_i)^p u(x)$ (with definitely chosen p) will reduce the equation to an equation of the same type, namely as in equation (1), but with the highest order of polynomial coefficients diminished by one, that is to say that now we have the case $k \to k - 1$. Thus, for the third-order case it simply leads to the second order, that allowed comprehensive analysis in terms of the equation of hypergeometric type [4].

We can now look for the solution of equation (28) corresponding to some low-order polynomials. Let us start looking for the first-order polynomial solution ($n = 1$). Then, the condition (3) reads $\beta = -\alpha$ and from (26) we obtain two possible nontrivial solutions ($c_n = 1$ as a normalization condition)

$$c_0^{(1)} = -x_4 \qquad \gamma^{(1)} = \alpha x_3 \tag{35}$$

$$c_0^{(2)} = -x_3 \qquad \gamma^{(2)} = \alpha x_4 \tag{36}$$

that gives for $Y_1(u)$

$$Y_1^{(1)}(u) = \exp\left\{-\frac{\alpha x_3x_4u^4}{64} + \frac{\alpha(x_3 + x_4)u^2}{8}\right\}\left(\frac{4}{u^2} - x_4\right)u^{\frac{3-2\alpha}{2}} \tag{37}$$

$$Y_1^{(2)}(u) = \exp\left\{-\frac{\alpha x_3x_4u^4}{64} + \frac{\alpha(x_3 + x_4)u^2}{8}\right\}\left(\frac{4}{u^2} - x_3\right)u^{\frac{3-2\alpha}{2}}. \tag{38}$$

It is evident that we have to put an additional requirement on the domain of the parameters α, x_3, x_4 to insure correct behaviour of the energy and of the wavefunctions. Without diving into technical difficulties (and losing significant features of the problem) it is better to consider the case when the appropriate conditions become maximally simple, so in what follows we will talk about the symmetric case when $x_3 = -x_4 = \xi > 0$. This leads to the vanishing of the fourth-order term in the potential (29) and as we said to simpler conditions imposed on the parameters of the allowed region.

Firstly, to obtain real spectra we must have $\xi \in \mathcal{R}$ so that the polynomial A_2 must have real roots. The vanishing of the wavefunction at infinity leads to the restriction on $\alpha < 0$, but we have to impose additionally the regularity condition at $u = 0$. At first glance we can impose the finiteness of the norm of the wavefunction, that gives $\alpha \leqslant -1/4$, but it is easy to see that the quantum potential in (30) is repulsive and infinite at $u = 0$ when the parameter $\alpha > -1/2$ or $\alpha \leqslant -3/2$ (after substitution of the value for β through α, of course). The latter means that we must put the boundary condition $Y(u)|_{u=0} = 0$ for the above-mentioned intervals, that leads to the $\alpha \leqslant -1/2$ requirement.

The constructed solutions, as one can see, are the two lowest bound states of the system; the eigenfunction (37) corresponds to the ground state and (38) to the first excited state for the potential

$$V(u) = \frac{\alpha^2 + 2\,\alpha + \frac{3}{4}}{u^2} + \frac{\alpha(\alpha-3)\xi^2u^2}{8} + \frac{\alpha^2\xi^4u^4}{256}. \tag{39}$$

In a similar way, for $n = 2$ we obtain the following three solutions for c_0, c_1, γ:

$$\begin{aligned} &c_0^{(1)} = -\frac{\alpha\xi^2}{\alpha+1} \qquad c_1^{(1)} = 0 \qquad \gamma^{(1)} = 0 \\ &c_0^{(\pm)} = \frac{\alpha\xi^2}{\alpha+2} \qquad c_1^{(\pm)} = \pm\frac{\xi\sqrt{2\alpha\,(2\,\alpha+3)}}{\alpha+2} \qquad \gamma^{(\pm)} = \pm\xi\,\sqrt{2\alpha\,(2\alpha+3)}. \end{aligned} \tag{40}$$

Then, the appropriate eigenfunctions are given by

$$Y^{(1)}(u) = \mathrm{e}^{\frac{\alpha u^4}{64}}\,u^{\frac{3}{2}-\alpha}\left(\frac{16}{u^4} - \frac{\alpha\xi^2}{\alpha+1}\right) \tag{41}$$

$$Y^{(\pm)}(u) = \mathrm{e}^{\frac{\alpha u^4}{64}}\,u^{\frac{3}{2}-\alpha}\left(\frac{16}{u^4} + \frac{\alpha\xi^2}{\alpha+2} \pm \frac{4\xi\sqrt{2\alpha\,(2\,\alpha+3)}}{(\alpha+2)u^2}\right). \tag{42}$$

Now, the admissible region for the parameter α is given by $\alpha \leqslant -5/2$. Then the constructed eigenstates represent the ground and the first two excited states for the potential

$$V(u) = \frac{\alpha^2\xi^4u^6}{256} - \frac{\alpha\xi^2\,(\alpha-3)\,u^2}{8} + \frac{4\alpha^2 + 24\,\alpha + 35}{4\,u^2}. \tag{43}$$

The construction of a higher-order polynomial solution can be easily continued but the main feature of the problem can be expressed already. As we saw, looking for the solution in the form of an nth-order polynomial and imposing relation (3) between α and β we obtain a specific quantum potential and for this potential the proposed methods allow us to construct just $n + 1$ eigenstates corresponding to different values of the parameter γ determined by the condition of vanishing matrix determinant. At least in some region of these free parameters (like α in the considered case) the last gives the allowed eigenstates starting from the ground and up to the nth excited state of the quantum system. At the same time, *all these eigenstates* are constructed on the polynomials of the same order, though the number of real roots differs from one polynomial to another. This is in evident contradistinction with the solvable case where the eigenfunctions for the nth excited state of the system correspond precisely to the nth-order polynomial having n real roots in the domain, and belonging to the same family of orthogonal polynomials.

It is relatively easy to check that the appropriate eigenfunctions we constructed are orthogonal, as they must be.

The main question which immediately emerges is how flexible are we in adjusting the parameters α, ξ, if we want to construct the solution for the equation of the form (26). So what should we do if we originally have the parameters A, B, C, D and ε as in (30)–(33) (for the symmetric case we consider $C = 0$ identically)? Let us try to express α etc through A, B, D, ε.

If we succeed in solving the equation (9) to construct polynomials and in obtaining the quantization condition for γ, then we obtain both the energy spectrum and the eigenfunctions of the bound states for the problem. Let us complete this program.

We will consider the equations (30)–(33) as the system of equations for the unknown α, ξ, satisfying the condition $\beta = -n\alpha - n(n-1)$. Then, equating the coefficients at all order monomials u^i for the potential $V(u)$ in (39) we obtain the system

$$-\tfrac{3}{4} + A + 2\,\alpha - \alpha^2 + 4\,((1-n)\,n - n\,\alpha) = 0 \tag{44}$$

$$B - \frac{3\,\alpha\,\xi^2}{8} + \frac{\alpha^2\,\xi^2}{8} = 0 \tag{45}$$

$$D - \frac{\xi^4\,\alpha^2}{256} = 0. \tag{46}$$

From the last equation we have for ξ^2, expressed through D, α,

$$\xi^2 = \pm 16\frac{\sqrt{D}}{\alpha}. \tag{47}$$

Then, after substituting the above into (45) we will have the following consistency condition for the parameters:

$$B = \pm\frac{2\,(\alpha - 3)}{\sqrt{D}}. \tag{48}$$

Solving the equation (44) for α we obtain

$$\alpha = 1 \pm \tfrac{1}{2}\sqrt{1 + 4\,A} - 2n. \tag{49}$$

Finally, substituting the expression for α into consistency condition (48) we rewrite it in a final form

$$B = 6\,\sqrt{D} - \sqrt{D}\,\left(2 \pm \sqrt{1 + 4\,A} - 4\,n\right). \tag{50}$$

So, having the definite parameters of the quantum potential A, B, D we have to satisfy only the one condition (50) to obtain the quasi-solvable potential with a definite number of bound states (namely n).

We consider now the next topologically different case when $A_3(x)$ has two different roots; evidently they have to be real. Again, we can place the first one at the origin, and, moreover, changing the scale of x we can put the second one at the position $x_1 = x_2 = 1$. Then $A_3(x) = x(x-1)^2$. The variable change reads $x = F(u) = \coth^2(u/2)$; the gauge transformation turns out to be

$$\begin{aligned}\chi(u) = -\frac{1}{4}(x_3 - 1)(x_4 - 1)\,\alpha\,\cosh\,u + \left(x_3\,x_4\,\alpha - \frac{1}{2}\right)\,\log\left(\cosh\,\frac{u}{2}\right)\\ + \left(\frac{3}{2} - \alpha\right)\,\log\left(\sinh\,\frac{u}{2}\right).\end{aligned} \tag{51}$$

Then we obtain the Schrödinger equation with the potential in the form

$$V(u) = \frac{1}{16}\sinh^4\frac{u}{2}\left[A + B\coth^2\frac{u}{2} + C\coth^4\frac{u}{2} + D\coth^6\frac{u}{2} + E\tanh^2\frac{u}{2}\right] \tag{52}$$

with the introduced parameters A, B, C, D, E expressed through the old ones as

$$\begin{aligned}A &= -8 + 32\alpha\,x_3\,x_4 - 8\alpha^2\,{x_3}^2\,x_4 - 8\alpha^2\,x_3\,{x_4}^2\\ B &= 2\big[\,5 - 4\,(-1 + 2\,x_4 + x_3\,(2 + 3\,x_4))\,\alpha\\ &\qquad + 2\left({x_3}^2 + 4\,x_3\,x_4 + {x_4}^2\right)\alpha^2 + 8\,n\,(-1 + n + \alpha)\,\big]\\ C &= -8\left(\alpha^2\,(x_3 + x_4) - 2\alpha\,(x_3 + x_4 - 2) + 1\right)\\ D &= (2\,\alpha + 1)\,(2\,\alpha + 3)\\ E &= (2\alpha\,x_3\,x_4 - 3)\,(2\alpha\,x_3\,x_4 - 1)\end{aligned} \tag{53}$$

where we have used the relation (3) between α and β.

In a similar way as we did in the previous case, one can proceed further to obtain some lower eigenstates for the problem with potential (52) and define the region of the parameters, allowing the existence of bound states. We do not intend to perform this routine task as the main features of such a class of the potentials are already exposed.

We discuss now the last topologically different cases when polynomial $A_3(x)$ has three different roots (we allow both real and complex roots for it).

As the formulas in this case become a little complicated, though the main features can be observed in some relatively simple examples, we concentrate on a most symmetric case, namely we assume that both roots of $A_3(x)$ and $A_2(x)$ are symmetric with respect to $x = 0$ (and those of $A_3(x)$ all are real). Then, again, we can move the position of the roots for $A_3(x)$ into 0, ± 1 and define the roots $A_2(x)$ as ξ.

Then the change of variables reads

$$u = \int_0^x \frac{\mathrm{d}v}{\sqrt{v(v^2-1)}} = -2\,F(-\mathrm{i}\,\mathrm{ArcSinh}(\sqrt{x}) \mid -1) \tag{54}$$

where $F(\phi \mid m)$ is an incomplete elliptic integral of the first kind [37], so that inverting it we have for $x = \mathcal{F}(u)$ (do not confuse $\mathcal{F}$ with F)

$$x = \mathcal{F}(u) = sn^2\left(\frac{u}{2} \mid -1\right) \tag{55}$$

where sn is an elliptic sine function [37].

The gauge transformation in this case can be represented implicitly by the following integral:

$$\chi(u) = \int_{u_0}^{u} \frac{1 - 2\,\alpha\,\xi^2 + (-3 + 2\,\alpha)\; sn^4(\frac{u}{2}, -1)}{4\,sn(\frac{u}{2}, -1)\sqrt{1 - sn^2(\frac{u}{2}, -1)}\sqrt{1 + sn^2(\frac{u}{2}, -1)}}\,\mathrm{d}u. \tag{56}$$

The quantum potential has the form

$$\begin{aligned} V(u) = {} & \frac{1}{16\,sn^2\left(\frac{u}{2} \mid -1\right)\left(sn^4\left(\frac{u}{2} \mid -1\right) - 1\right)} \bigg\{ -3 + 8\,\alpha\,\xi^2 - 4\,\alpha^2\,\xi^4 \\ & + 2\,(-3 - 8\;((1-n)\;n - n\,\alpha) + 4\,\alpha^2\,\xi^2 + \alpha\,(4 - 12\,\xi^2))\,sn^4\left(\frac{u}{2} \mid -1\right) \\ & + (-3 + 8\,\alpha - 4\,\alpha^2 + 16\;((1-n)\;n - n\,\alpha))\,sn^8\left(\frac{u}{2} \mid -1\right)\bigg\}. \end{aligned} \tag{57}$$

As the elliptic functions are periodic, we obtain a family of quasi-solvable potentials which generalizes in some sense the exactly solvable Pöschl–Teller potentials [1]. Moreover, similar to Pöschl–Teller potentials, this family of potentials evidently does not correspond to any band structure, because the potential, being a periodic one, has singular points that impose a zero-value boundary condition on the wavefunctions there.

The potentials of the given class are represented in figures 1–3 for some specific values of the parameters α, ξ and n to demonstrate the richness of their behaviour.

The degenerate cases when $A_3(x)$ is reduced to a lower-order polynomial as we know from [4] could also be interesting, but we do not intend to consider them here but in a separate publication.

5. Discussion

To summarize, we implement the method proposed by us earlier for the investigation of the quasi-solvability of some quantum problems which can be transformed to the SODE

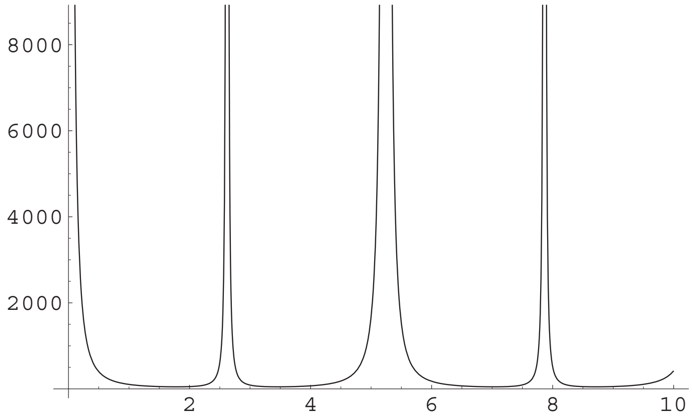

Figure 1. Quantum potential $V(u)$ in (57) with the following parameter values: $\alpha = -1,\ \xi = 3,\ n = 1$.

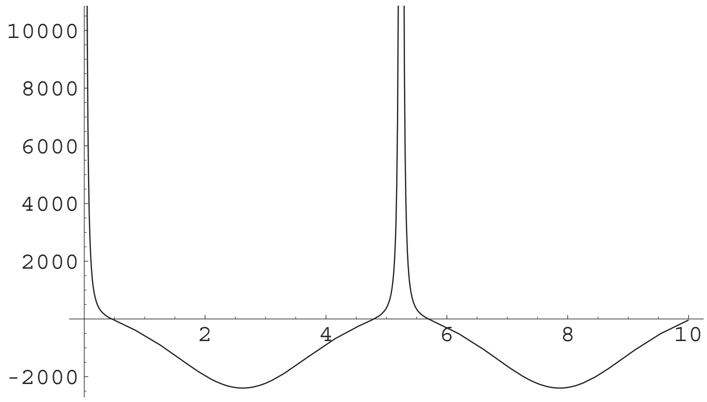

Figure 2. Quantum potential $V(u)$ in (57) with the following parameter values: $\alpha = -1,\ \xi = 2,\ n = 50$.

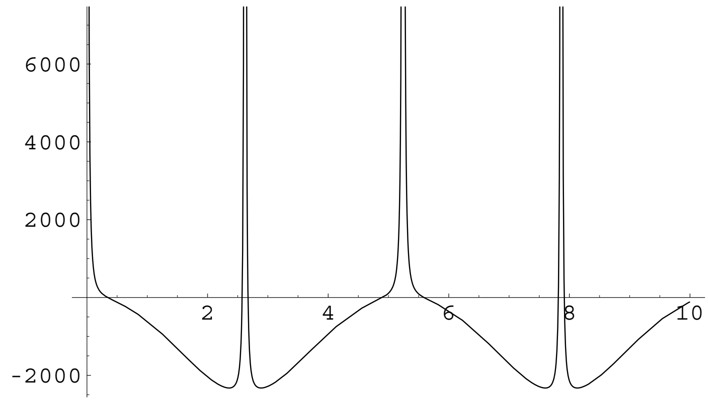

Figure 3. Quantum potential $V(u)$ in (57) with the following parameter values: $\alpha = 1,\ \xi = 2,\ n = 50$.

possessing polynomial solutions, considering in detail the case of the third-order polynomial as a coefficient function. In contrast to the solvable case all eigenstates constructed here are polynomials of the same order, whereas the number of their real roots varies from zero up to the number of the eigenstates allowed to be constructed algebraically.

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References

[1] Grosche C and Steiner F 1998 *Handbook of Feynman Path Integrals* vol STMP 145 (Berlin: Springer)
[2] Landau L D and Lifshitz E M 1989 *Quantum Mechanics* 3rd edn (Oxford: Reed)
[3] Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* vol 1–2 (New York: McGraw-Hill)
[4] Krylov G and Robnik M 2000 On 1D Schrödinger problems allowing polynomial solutions *J. Phys. A: Math. Gen.* **33** 1233–45
[5] Natanzon G A 1971 *Vestn. LGU* 22–8 (in Russian)
[6] Natanzon G A 1979 *Teor. Mat. Fiz.* **38** 219–29 (in Russian)
[7] Turbiner A 1992 Lie algebras and polynomials in one variable *J. Phys. A: Math. Gen.* **25** L1087–93
[8] Bhattacharjee A and Sudarshan E C G 1962 A class of solvable potentials *Nuovo Cimento* **25** 864–79
[9] Infeld L and Hull T E 1951 The factorization method *Rev. Mod. Phys.* **23** 21–68
[10] Witten E 1981 Dynamical breaking of supersymmetry *Nucl. Phys.* B **185** 513–54
[11] Delaney D and Nieto M M 1990 SUSY-WKB is neither exact nor never worse than WKB for all solvable potentials *Phys. Lett.* B **247** 301–7
[12] Junker G 1996 *Supersymmetric Methods in Quantum and Statistical Physics* (Berlin: Springer)
[13] Andrianov A A *et al* 1997 Matrix Hamiltonians: SUSY approach to hidden symmetries *J. Phys. A: Math. Gen.* **30** 5037–50
[14] Dunne G V and Feinberg J 1998 Self-isospectral periodic potentials and supersymmetric quantum mechanics *Phys. Rev.* D **57** 1271–6
[15] Choi Je-Young and Hong Seok-In 1998 Supersymmetric quantum mechanics with nonlocal potentials *Phys. Rev.* A **60** 796
[16] Andrianov A *et al* 1999 SUSY quantum mechanics with complex superpotentials and real energy spectra *Int. J. Mod. Phys.* A **14** 2675–88
[17] Das T K and Chakrabarti B 1999 Application of supersymmetry to a coupled system of equations: the concept of a superpotential matrix *J. Phys. A: Math. Gen.* **32** 2387–94
[18] Tkachuk V M 1999 Supersymmetric method for constructing quasi-exactly and conditionally-exactly solvable potentials *J. Phys. A: Math. Gen.* **32** 1291–300
[19] Olshanetsky M A and Perelomov A M 1983 Quantum integrable systems related to Lie algebras *Phys. Rep.* **94** 313–404
[20] Alhassid Y, Gursey F and Iachello F 1983 Group theory approach to scattering *Ann. Phys., NY* **148** 346–80
[21] Kakei S 1997 Intertwining operators for a degenerate double affine hecke algebra and multivariable orthogonal polynomials *Preprint* q-alg/970601
[22] Horozov E and Kasman A 1998 Darboux transformations of bispectral quantum integrable systems *Preprint* math-ph/9806002
[23] Quesne C 1999 Comment on application of nonlinear deformation algebra to a physical system with Pöschl–Teller potential *J. Phys. A: Math. Gen.* **32** 6705–10
[24] Beckers Y, Brihaye Y Y and Debergh N 1999 On realizations of nonlinear Lie algerbas by differential operators *J. Phys. A: Math. Gen.* **32** 2791–803
[25] Krähmer D S, Schleich W P and Yakovlev V P 1998 Confined quantum systems: the parabolically confined hydrogen atom *J. Phys. A: Math. Gen.* **31** 4493–520
[26] Bender C M *et al* 1999 Complex square well—a new exactly solvable quantum mechanical model *J. Phys. A: Math. Gen.* **32** 6771–81
[27] Robnik M and Romanovski V G 1999 On Urabe's criteria of isochronicity *J. Phys. A: Math. Gen.* **32** 1279–83

[28] Gangopadhyaya A and Khare A 1995 Methods for generating quasi-exactly solvable potentials *Phys. Lett.* A **208** 261–8

[29] Khare A and Mandal B P 1998 Do quasi-exactly solvable systems always correspond to orthogonal polynomials? *Phys. Lett.* A **239** 197–200

[30] Khare A and Sukhatme U 1999 New solvable and quasi exactly solvable periodic potentials *J. Math. Phys.* **40** 5473–94

[31] Cicuta G M and Ushveridze A G 1995 Quasi-exactly solvable problems and random matrix theory *Preprint* hep-th/9510057

[32] Krajewska A, Ushveridze A and Walczak Z 1996 On the duality of quasi-exactly solvable problems *Preprint* hep-th/9601025

[33] Krajewska A, Ushveridze A and Walczak Z 1996 Bender–Dunne orthogonal polynomials and quasi-exact solvability *Preprint* hep-th/9601088

[34] Bender C M and Boettecher S 1998 Quasi-exactly solvable quartic potential *J. Phys. A: Math. Gen.* **31** L273–7

[35] Bender C M and Dunne G V 1996 Quasi-exactly solvable systems and orthogonal polynomials *J. Math. Phys.* **37** 6

[36] Shifman M and Turbiner A 1999 Energy reflection symmetry of lie-algebraic problems: where the quasiclassical and weak coupling expansions meet *Phys. Rev.* A **59** 1791

[37] Gradshteyn I S and Ryzhik I M 1994 *Table of Integrals, Series and Products* (Boston: Academic)